Solution Sheet 3

Exercise 3.1.

Let $(\mathcal{X}_i)$ be a collection of separable metric spaces. Prove that the infinite product $\Pi_{i=1}^{\infty}\mathcal{X}_i$, equipped with product topology, is separable.

Proof. Recall that a set A is dense in $\Pi_{i=1}^{\infty}\mathcal{X}_i$ if for every non-empty open set $O \subset \Pi_{i=1}^{\infty}\mathcal{X}_i$, we have that $A \cap O \neq \emptyset$. Also recall that, from the definition of the product topology, if $O \subset \Pi_{i=1}^{\infty}\mathcal{X}_i$ is non-empty and open then there exists an open set $U \subset O$ such that

$$U = (\Pi_{i=1}^{n} U_i) \times \left(\Pi_{i=n+1}^{\infty}\mathcal{X}_i\right) \tag{1}$$

where each $U_i \subset \mathcal{X}_i$ is open. Let D_i be the countable dense subset of $\mathcal{X}_i$, and fix any sequence of elements (x_i) such that $x_i \in \mathcal{X}_i$. Define

$$A_n := (\Pi_{i=1}^{n} D_i) \times \left(\Pi_{i=n+1}^{\infty}\{x_i\}\right)$$

and

$$A := \bigcup_{n=1}^{\infty} A_n.$$

Then A is countable as the countable union of countable sets, and it is dense as any set of the form (1) has non-zero intersection with $A_n \subset A$ by the density of each D_i in $\mathcal{X}_i$, $i = 1, \ldots, n$.

□

Exercise 3.2.

Let (x_n) be a sequence in the metric space $\mathcal{X}$, and $x \in \mathcal{X}$. Prove that (x_n) converges to x in $\mathcal{X}_n$ if and only if the associated sequence of delta measures (δ_{x_n}) converges weakly to δ_x.

Proof. We consider the implications separately:

$\Longrightarrow$: We assume that $x_n \to x$, and wish to show that for all bounded and continuous $f : \mathcal{X} \to \mathbb{R}$ that

$$\int_{\mathcal{X}} f d\delta_{x_n} \to \int_{\mathcal{X}} f d\delta_x$$

which is by definition simply $f(x_n) \to f(x)$. This is immediate from the continuity of f.

$\Longleftarrow$: We assume that, as described above, $f(x_n) \to f(x)$ for all bounded and continuous f, and wish to show that $x_n \to x$. Suppose for a contradiction that this was not true; thus, there exists an $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists an $n > N$ such that $x_n \notin B_\varepsilon(x)$, where $B_\varepsilon(x)$ denotes the open ball of radius ε centred at x. We can construct a bounded continuous function f, exactly as we did in Exercise 2.4, satisfying $f(x) = 1$ and $f(z) = 0$ for all $z \notin B_\varepsilon(x)$. Then $f(x_n)$ cannot converge to $f(x)$, generating the desired contradiction.

□

Exercise 3.3.

Let $\mathcal{X}$ be a separable metric space, and (μ_n) be a collection of (uniformly) tight probability measures on $\mathcal{X}$.

1. Show that there exists a subsequence (μ_{n_k}) and a probability measure μ on $\mathcal{X}$ such that (μ_{n_k}) is weakly convergent to μ.

2. Let $\varepsilon > 0$ and K be a compact set such that $\mu_n(\mathcal{X}\backslash K) < \varepsilon$ for all n. Prove that $\mu(\mathcal{X}\backslash K) \leq \varepsilon$.

Proof. We prove the parts in turn:

1. This is a direct application of Prohorov's Theorem (Theorem 2.3.7) and the property of being relative compact in the topology of weak convergence in $\mathbb{P}(\mathcal{X})$.

2. As K is compact then it is closed, so $\mathcal{X}\backslash K$ is open. By the Portmanteau Theorem (Proposition 2.3.2) then
$$\mu(\mathcal{X}\backslash K) \leq \liminf_{n_k \to \infty} \mu_{n_k}(\mathcal{X}\backslash K).$$
By assumption each $\mu_{n_k}(\mathcal{X}\backslash K) < \varepsilon$, so $\liminf_{n_k\to\infty} \mu_{n_k}(\mathcal{X}\backslash K) \leq \varepsilon$, hence the result.

□

Exercise 3.4.

Let $\mathcal{X}$ be a metric space and $\mathbb{P}(\mathcal{X})$ the space of probability measures on $\mathcal{X}$. Prove that limits in the topology of weak convergence on $\mathcal{X}$ are unique.

Proof. Suppose that (μ_n) is a sequence in $\mathbb{P}(\mathcal{X})$ which is weakly convergent to both μ and ν. Then for every bounded and continuous function $f : \mathcal{X} \to \mathbb{R}$ we have that $\left(\int_{\mathcal{X}} f d\mu_n\right)$ converges to both $\int_{\mathcal{X}} f d\mu$ and $\int_{\mathcal{X}} f d\nu$. By uniqueness of limits in $\mathbb{R}$ then
$$\int_{\mathcal{X}} f d\mu = \int_{\mathcal{X}} f d\nu$$
which implies that $\mu = \nu$ due to Theorem 2.2.2. □

Exercise 3.5.

Let $\mathcal{X}$, $\mathcal{Y}$ be metric spaces and (μ_n) a sequence in $\mathbb{P}(\mathcal{X})$, (ν_n) a sequence in $\mathbb{P}(\mathcal{Y})$. Prove that the sequence of product measures $(\mu_n \otimes \nu_n)$ is tight in $\mathbb{P}(\mathcal{X} \times \mathcal{Y})$ if and only if (μ_n) is tight in $\mathbb{P}(\mathcal{X})$ and (ν_n) is tight in $\mathbb{P}(\mathcal{Y})$.

Proof. We prove the implications in turn:

$\Longrightarrow$: We assume that $(\mu_n \otimes \nu_n)$ is tight. Then for any given $\varepsilon > 0$ which is henceforth fixed, there exists a compact $K \subset \mathcal{X} \times \mathcal{Y}$ such that $\mu_n \otimes \nu_n(K) > 1 - \varepsilon$ for all $n \in \mathbb{N}$. To prove tightness of (μ_n) we wish to find a compact $K_X \subset \mathcal{X}$ such that $\mu_n(K_X) > 1 - \varepsilon$ for all $n \in \mathbb{N}$. We define $K_X := \pi_X(K)$ where π_X is the projection mapping onto X. As π_X is continuous with respect to the product topology, then $\pi_X(K)$ is the continuous image of a compact set hence compact. Moreover,
$$\mu_n(\pi_X(K)) = \mu_n \otimes \nu_n\left((\pi_X(K) \times \mathcal{Y}\right) \geq \mu_n \otimes \nu_n\left(K\right) > 1 - \varepsilon$$
as required. The same process holds true for (ν_n), completing the proof.

$\Longleftarrow$: We assume that (μ_n) and (ν_n) are tight. Then for any given $\varepsilon > 0$ which is henceforth fixed, there exists compact sets $K_X \subset \mathcal{X}$ and $K_Y \subset \mathcal{Y}$ such that $\mu_n(K_X) > 1-\varepsilon$, $\nu_n(K_Y) > 1-\varepsilon$ for all $n \in \mathbb{N}$. We wish to find a compact $K \subset \mathcal{X} \times \mathcal{Y}$ such that $\mu_n \otimes \nu_n(K) > 1-\varepsilon$, or equivalently that $\mu_n \otimes \nu_n(\mathcal{X} \times \mathcal{Y} \backslash K) < \varepsilon$. Define $K := K_X \times K_Y$, which is compact as the product of compact sets in the product topology. Observe that

$$K = (K_X \times \mathcal{Y}) \cap (\mathcal{X} \times K_Y)$$

so using the general rule that $\mathbb{P}(A \cap B) \geq \mathbb{P}(A) + \mathbb{P}(B) - 1$, then

$$\begin{aligned} \mu_n \otimes \nu_n(K) &\geq \mu_n \otimes \nu_n(K_X \times \mathcal{Y}) + \mu_n \otimes \nu_n(\mathcal{X} \times K_Y) - 1 \\ &= \mu_n(K_X) + \nu_n(K_Y) - 1 \\ &> (1-\varepsilon) + (1-\varepsilon) - 1 \\ &> 1 - 2\varepsilon \end{aligned}$$

which is sufficient to conclude by an arbitrary replacement of ε with $\frac{\varepsilon}{2}$.

□

Exercise 3.6.

Let $(\mu_n), \mu \in \mathbb{P}(\mathbb{R}^d)$ satisfy that for all continuous and compactly supported $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_{\mathbb{R}^d} f d\mu_n \longrightarrow \int_{\mathbb{R}^d} f d\mu \tag{2}$$

as $n \to \infty$. Assume in addition that the sequence (μ_n) is tight. Prove that (μ_n) is weakly convergent to μ.

Proof. We fix an arbitrary continuous and bounded $\phi : \mathbb{R}^d \to \mathbb{R}$, with the aim to show that

$$\int_{\mathbb{R}^d} \phi d\mu_n \longrightarrow \int_{\mathbb{R}^d} \phi d\mu$$

as $n \to \infty$. Recall the following classical fact from analysis: if (a_n) is a bounded sequence in $\mathbb{R}$ such that every convergent subsequence is convergent to $a \in \mathbb{R}$, then the full sequence (a_n) is convergent to a. We set

$$a_n := \int_{\mathbb{R}^d} \phi d\mu_n.$$

Firstly each $|a_n|$ is bounded by $\sup_{x \in \mathbb{R}^d} |\phi(x)|$. Take any convergent subsequence $a_{n_k} \to a$. As (μ_{n_k}) is tight then it contains a further subsequence (μ_{m_j}) which is weakly convergent to some $\nu \in \mathbb{P}(\mathbb{R}^d)$. By definition of this weak convergence,

$$\int_{\mathbb{R}^d} \phi d\mu_{m_j} \longrightarrow \int_{\mathbb{R}^d} \phi d\nu$$

and the limit a of (a_{n_k}) must agree with the limit of its subsequence (a_{m_j}), so $a = \int_{\mathbb{R}^d} \phi d\nu$. We now show that $\nu = \mu$. Indeed as (μ_{m_j}) is weakly convergent to μ then certainly for all continuous and compactly supported $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_{\mathbb{R}^d} f d\mu_{m_j} \longrightarrow \int_{\mathbb{R}^d} f d\nu.$$

Combined with (2),

$$\int_{\mathbb{R}^d} f d\nu = \int_{\mathbb{R}^d} f d\mu$$

for all such f. From the solution of Exercise 2.4 this implies that $\mu = \nu$ as desired, so

$$a = \int_{\mathbb{R}^d} \phi d\mu.$$

This limit is independent of the choice of converging subsequence, so every convergent subsequence of (a_n) gives a, hence the entire sequence converges to a which concludes the proof.

□

Exercise 3.7.

Let $\mathcal{X}, \mathcal{Y}$ be complete and separable metric spaces, (μ_n) a sequence in $\mathbb{P}(\mathcal{X})$ weakly convergent to some $\mu \in \mathbb{P}(\mathcal{X})$ and (ν_n) a sequence in $\mathbb{P}(\mathcal{Y})$ weakly convergent to some $\nu \in \mathbb{P}(\mathcal{Y})$. Prove that the sequence of product measures $(\mu_n \otimes \nu_n)$ is weakly convergent to $\mu \otimes \nu$ in $\mathbb{P}(\mathcal{X} \times \mathcal{Y})$.

Proof. We look to take a similar approach to the proof of Exercise 3.6. Indeed Exercise 3.6 relied on the fact that the convergence (2) was known for a measure determining class of functions f, and that the sequence of measures was tight. For the first of these ingredients, we consider the class of all functions $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ such that there exists a continuous bounded $g : \mathcal{X} \to \mathbb{R}$ and $h : \mathcal{Y} \to \mathbb{R}$ whereby $f(x, y) = g(x)h(y)$. We claim two properties:

1. For all such f,

   $$\int_{\mathcal{X}\times\mathcal{Y}} f d(\mu_n \otimes \nu_n) \longrightarrow \int_{\mathcal{X}\times\mathcal{Y}} f d(\mu \otimes \nu)$$

   as $n \to \infty$.

2. This class of functions is measure determining.

For the first claim, observe that

$$\int_{\mathcal{X}\times\mathcal{Y}} f d(\mu_n \otimes \nu_n) = \left(\int_{\mathcal{X}} g d\mu_n\right)\left(\int_{\mathcal{Y}} h d\nu_n\right)$$

by Fubini's Theorem, and similarly for $\mu \otimes \nu$. By the weak convergence assumption we have that

$$\int_{\mathcal{X}} g d\mu_n \longrightarrow \int_{\mathcal{X}} g d\mu$$

and similarly for $\int_{\mathcal{Y}} h d\nu_n$, from which we deduce the first claim. For the second we invoke Theorem 2.2.5, as $\mathcal{X} \times \mathcal{Y}$ is itself a complete separable metric space, and the class of such functions is an algebra as $(g_1 h_1)(g_2 h_2) = (g_1 g_2)(h_1 h_2)$ is in the class, and separates points as one can take g or h to be the constant 1 as in Exercise 2.6. Theorem 2.2.5 thus gives the second claim.

The other aforementioned ingredient is that the sequence of measures $(\mu_n \otimes \nu_n)$ is tight. This arises from showing that (μ_n) is tight in $\mathbb{P}(\mathcal{X})$ and similarly for (ν_n), appealing to the converse of Prohorov's Theorem, Theorem 2.3.8; indeed (μ_n) weakly convergent to μ implies that every subesequence has a further subsequence weakly convergent to μ by Proposition 2.3.3, hence the collection (μ_n) is relatively compact so tight by the converse of Prohorov's Theorem. The same is true of (ν_n), hence the product is tight by Exercise 3.5.

With these properties, the proof is identical to that of Exercise 3.6.

□